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Stereogenicity/astereogenicity as global/local permutation-group symmetry and relevant concepts for restructuring stereochemistry

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Molecules of ligancy 4 that have been derived from an allene, an ethylene, a tetrahedral, and a square-planar skeleton have been investigated to show that their symmetries are dually and distinctly controlled by point groups and permutation groups. Insomuch as the point-group symmetry was exhibited to control the chirality/achirality of a molecule, sphericity in a molecule, and enantiomeric relationship between molecules [S. Fujita, J. Am. Chem. Soc. 112 (1990) 3390], the permutation-group symmetry has been now clarified to control the stereogenicity of a molecule, tropicity in a molecule, and diastereomeric relationship between molecules. To characterize permutation groups, proper and improper permutations have been defined by comparing proper and improper rotations. Thereby, such permutation groups are classified into stereogenic and astereogenic ones. After a coset representation (CR) of a permutation group has been ascribed to an orbit (equivalence class), the tropicity of the orbit has been defined in term of the global stereogenicity and the local stereogenicity of the CR. As a result, the conventional "stereogenicity" has now been replaced by the concept *local stereogenicity* of the present investigation. The terms *homotropic*, *enantiotropic*, and *hemitropic* are coined and used to characterize prostereogenicity. Thus, a molecule is defined as being prostereogenic if it has at least one enantiotropic orbit. Since this definition has been found to be parallel with the definition of prochirality, relevant concepts have been discussed with respect to the parallelism between stereogenicity and chirality in order to restructure the theoretical foundation of stereochemistry and stereoisomerism. The derivation of the skeletons has been characterized by desymmetrization due to the subduction of CRs. The Cahn–Ingold–Prelog (CIP) system has been discussed from the permutational point of view to show that it specifies diastereomeric relationships only. The apparent specification of enantiomeric relationships by the CIP system has been shown to stem from the fact that diastereomeric relationships and enantiomeric ones overlap occasionally in case of tetrahedral molecules.

KEY WORDS: stereochemistry, stereogenicity, chirality, permutation group, diastereomer

1. Introduction

Organic [1,2] and inorganic stereochemistry [3] have had such a common standpoint that they have been based on polyhedral or polygonal models. However, because the two disciplines have been developed distinctly, concepts proposed precedently for organic stereochemistry have not been so fully applied to inorganic stereochemistry. In order to remedy this situation, it is necessary to create a new framework for covering both of the disciplines after the scope and limitations of the precedent concepts are reinvestigated critically.

From its beginning, organic stereochemistry has been mainly based on a tetrahedral model proposed by van't Hoff [4,5], where the term "asymmetric carbon center" has been a key concept of characterizing stereoisomers. However, the term seriously confused chemists and biochemists, because it does not always correspond to the chirality of a molecule, as pointed out in reviews [6,7] and textbooks [1,2]. The sequence rule proposed by Cahn, Ingold and Prelog (the CIP system) [8] and its application to the "prochirality" by Hanson [9] revived the confusion due to "asymmetric carbon center" and relevant terms. Mislow and Siegel [10] gave a convincing argument to remedy the confusion and recommended the terms "stereogenic center" and "prostereogenic center" proposed earlier by McCasland [11] in place of such terms as "chiral center" and "prochiral center". They pointed out the importance of local chirality so as to propose the term "chirotopic"; then they emphasized that chirotopicity and stereogenicity are conceptually distinct. The revised CIP system adopted the recommendation in the form of the descriptors of stereogenic units etc. [12–15]. However, the adoption of the terms "stereogenic" and "prostereogenic" in place of the "chirality" of the original CIP system [8] and Hanson's "prochirality" [9] brought some apparent settlement of the confusion, because the mathematical or logical meanings of the terms were not clarified. In particular, the question *how chirality and stereogenicity are distinct* has not been fully investigated.

The concepts of chirality and prochirality have been comprehended by virtue of the concept of sphericity which we have previously proposed [16]. The concept of sphericity, which is based on a coset representation (CR) and of its subduction [17], has been applied to the redefinition of prochirality [16], topicity [18,19],[1] stereogenicity [19], and anisochrony [21]. It has also given a sound foundation to accomplish isomer enumeration [17–26].[2]

On the other hand, few efforts to clarify the concepts of "stereogenicity" and "prostereogenicity" have been conducted in spite of the practical success of the CIP system. We have discussed "chirality" and "stereogenicity" for a tetrahedral model in terms of the observance and violation of chirality fittingness [30]. To solve the question *how chirality and stereogenicity are distinct*, however, we should further pursue the meanings

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[1] The sphericity terms describe an attribute of an orbit (equivalence class) [16]. In contrast, the topicity terms are concerned with two categories of different contexts, i.e., a relationship between two sites [20] and an attribute of each site [10]. We use the sphericity terms and do not use the topicity terms throughout this paper. The term *homospheric* corresponds to the combination of "homotopic and achirotopic". The term *enantiospheric* corresponds to the combination of "enantiotopic and chirotopic". The term *hemispheric* corresponds to the combination of "homotopic and chirotopic". The term "diastereotopic" means the difference in orbits.

[2] For reviews on combinatorial enumeration, see [18,27–29].

Table 1
Parallel concepts for point-group and permutation-group symmetries.

Point-group symmetry[a]	Permutation-group symmetry[b]
chiral	stereogenic
achiral	astereogenic
proper rotation	proper permutation
improper rotation	improper permutation
coset representation (CR)	coset representation (CR)
orbit (equivalence class)	orbit (equivalence class)
global (point-group) symmetry	global permutation-group symmetry
local (point-group) symmetry	local permutation-group symmetry
local chirality	local stereogenicity
sphericity: enantiospheric	tropicity: enantiotropic
homospheric	homotropic
hemipheric	hemitropic
chirality fittingness	stereogenicity fittingness
prochiral	prostereogenic
enantiomeric	diastereomeric

[a] See [16,18].
[b] The present results.

of “stereogenicity” and “prostereogenicity” on the same level as our comprehension on “chirality” and “prochirality” has reached.

In this paper, we will give a new definition of *stereogenicity*, which is conceptually different from, but consistent with the original meaning of “stereogenicity”. Thereby, the nature of stereogenecity/astereogenicity will be ascribed to permutation-group symmetry insomuch as the nature of chirality/achirality is ascribed to point-group symmetry. We will emphasize the importance of orbits (equivalence classes) generated by the actions of point groups and permutation groups. We will give a new meaning to diastereomeric relationship independently of enantiomeric relationship. Furthermore, we will propose various related concepts for restructuring stereochemistry, e.g., stereoisomeric groups, proper and improper permutations, stereogenic and astereogenic groups, the maximal stereogenic subgroup, local permutation-group symmetry and local stereogenicity, as well as tropic terms (homotropic, enantiotropic, hemitropic). These concepts are discussed in terms of parallelism between point-group and permutation-group symmetries (table 1).

To show the wide applicability of the present treatment, we will examine organic and inorganic molecules of ligancy 4 that are listed in figure 1, i.e., allenes, ethylenes, methanes, and square-planar complexes (A: achiral ligand; and M: metal).[3] Under point-group symmetry, each of the molecules is considered to be derived by placing an appropriate set of four ligands on the substitution positions of an achiral skeleton, i.e., an

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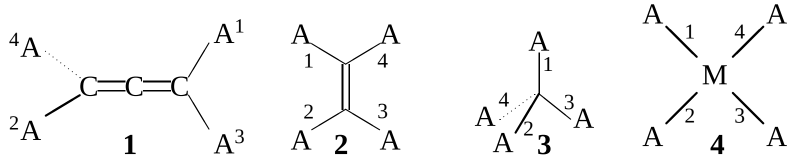
[3] Strictly speaking, the terms “proligand” and “promolecule” should be used in place of the terms “ligand” and “molecule” [31]. However, the latter terms are used in the present paper, since such usage may cause no confusion within the scope of the present discussion.



CR of point-group type:

$D_{2d}(/C_s)$ $D_{2h}(/C''_s)$ $T_d(/C_{3v})$ $D_{4h}(/C''_{2v})$

CR of permutation-group type:

$S_9^{[4]}(/S_3^{[4]})$ $S_9^{[4]}(/S_{2'}^{[4]})$ $S^{[4]}(/S_8^{[4]})$ $S^{[4]}(/S_8^{[4]})$

Figure 1. Molecules of ligancy 4.

allene skeleton of D_{2d}-symmetry, an ethylene skeleton of D_{2h}-symmetry, a tetrahedral skeleton of T_d-symmetry, or a square-planar skeleton of D_{4h}-symmetry. The detailed investigation of their permutation-group symmetries provides us a deeper insight into stereochemistry and stereoisomerism.

2. Stereoisomeric relationships revisited

2.1. Overlap of enantiomeric and diastereomeric relationships

To begin with, it is worthwhile to mention the misleading nature of a conventional terminology for stereoisomerism, where the importance of the differentiation between isomeric relationships and the resulting isomers is usually underestimated.[4] Stereoisomers are conventionally defined as isomers that have the same constitution but differ only in configuration and/or conformation. Stereoisomers are subdivided into enantiomers and diastereomers by virtue of the conventional terminology. Thus, enantiomers are first defined as a pair of a stereoisomer and its mirror image that is not superimposable to the original stereoisomer. After this definition, diastereomers are defined as stereoisomers that are not enantiomers.

Rigorously speaking, the statement "Stereoisomers are subdivided into enantiomers and diastereomers" is correct only if we consider a dichotomous classification scheme of stereoisomeric relationships [32]. It is incorrect if we take account of the division of a set of stereoisomers. In fact, such a set of stereoisomers is subdivided into *several pairs of enantiomers*, but, by no means, into *enantiomers and diastereomers*. A pair and another pair of enantiomers, which are selected from the resulting

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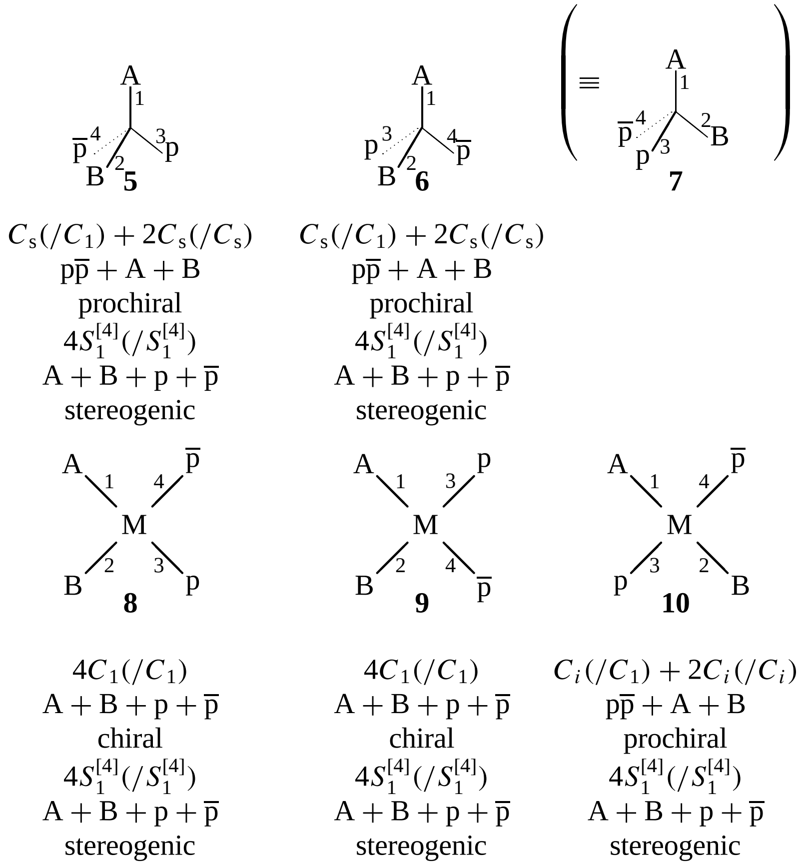
[4] In general, an *isomeric relationship* is determined by a procedure of pairwise comparison between two objects. Such procedures are applied to every pairs of objects to produce a set of equivalent objects. The set is an equivalence class, which is called *an orbit*. The members of the orbit are called *isomers*. Each isomeric relationship has its criterion to equalify a pair of objects. One of the aims of the present paper is to clarify that the criterion for enantiomeric relationship is point-group symmetry while the criterion for diastereomeric relationship is permutation-group symmetry. Another aim is to clarify how two relationships are distinct. In the present paper, the concepts of orbits as equivalence classes under permutation-group symmetry as well as under point-group symmetry are taken into primary consideration.



5
$C_{\mathrm{s}}(/C_1) + 2C_{\mathrm{s}}(/C_{\mathrm{s}})$
$\mathrm{p}\overline{\mathrm{p}} + \mathrm{A} + \mathrm{B}$
prochiral
$4S_1^{[4]}(/S_1^{[4]})$
$\mathrm{A} + \mathrm{B} + \mathrm{p} + \overline{\mathrm{p}}$
stereogenic

6
$C_{\mathrm{s}}(/C_1) + 2C_{\mathrm{s}}(/C_{\mathrm{s}})$
$\mathrm{p}\overline{\mathrm{p}} + \mathrm{A} + \mathrm{B}$
prochiral
$4S_1^{[4]}(/S_1^{[4]})$
$\mathrm{A} + \mathrm{B} + \mathrm{p} + \overline{\mathrm{p}}$
stereogenic

(≡ **7**)

8
$4C_1(/C_1)$
$\mathrm{A} + \mathrm{B} + \mathrm{p} + \overline{\mathrm{p}}$
chiral
$4S_1^{[4]}(/S_1^{[4]})$
$\mathrm{A} + \mathrm{B} + \mathrm{p} + \overline{\mathrm{p}}$
stereogenic

9
$4C_1(/C_1)$
$\mathrm{A} + \mathrm{B} + \mathrm{p} + \overline{\mathrm{p}}$
chiral
$4S_1^{[4]}(/S_1^{[4]})$
$\mathrm{A} + \mathrm{B} + \mathrm{p} + \overline{\mathrm{p}}$
stereogenic

10
$C_i(/C_1) + 2C_i(/C_i)$
$\mathrm{p}\overline{\mathrm{p}} + \mathrm{A} + \mathrm{B}$
prochiral
$4S_1^{[4]}(/S_1^{[4]})$
$\mathrm{A} + \mathrm{B} + \mathrm{p} + \overline{\mathrm{p}}$
stereogenic

Figure 2. Enantiomeric and diasteromeric relationships overlapped.

pairs of enantiomers, are regarded as being diastereomeric. Hence, the original statement for the conventional terminology should be revised into the following preferred one: "Stereoisomeric relationship is subdivided into enantiomeric relationship and diastereomeric relationship." This terminology of classification implies that enantiomeric relationship is regarded as being superior to diastereomeric relationship.

By keeping this discussion in mind, we can say that enantiomeric relationship generates such pairs of enantiomers as orbits (equivalence classes). However, diastereomeric relationship does not generate any orbits, so long as it is defined as a subsidiary concept, as we have seen in the conventional terminology for stereoisomerism. For example, let us examine tetrahedral molecules **5** and **6**, both of which have achiral ligands (A and B) and a pair of enantiomeric ligands (p and $\overline{\mathrm{p}}$). Under point-group symmetry, they are achiral because each of them coincides with itself. The conventional way to recognize the diastereomeric relationship between **5** and **6** is to test the nature of being *not* enantiomeric, where the procedure of the recognition does not stem from such direct operations as point-group operations.

An alternative way for specifying diastereomeric relationship is to test the permutability of ligands, which is based on direct operations such as permutations [33,34].[5]

For example, the permutation between p(3) and $\overline{\mathrm{p}}(4)$ in **5**, which is represented by a cycle product (1)(2)(3 4), generates the corresponding diasteromer **6**. Thus, the permutability can be a clue of specifying the diastereomeric relationship between the tetrahedral molecules **5** and **6**. If we take account of permutability, square-planar complexes **8**, **9**, and **10** are diastereomeric, because they are interconvertible into each other. Note that the permutation (1)(2)(3 4) for the conversion between the former two complexes **8** to **9** is the same as the one for converting **5** into **6**. Logically speaking, we should say that the square-planar complexes **8** and **9** are diastereomeric by the same reason as we recognize that the tetrahedral molecules **5** and **6** are diastereomeric. In contrast, **8** and **9** are alternatively determined to be enantiomeric under point-group symmetry. Hence, **8** and **9** are concluded to be enantiomeric and diastereomeric at the same time. Since the square-planar complex **8** is converted into **10** the permutation (1)(2 3)(4), they are decided to be diastereomeric. Obviously, this diastereomeric relationship does not overlap enantiomeric one. Note that the same permutation (1)(2 3)(4) converts the tetrahedral molecule **5** into **7**, which is homomeric to (or identical with) **6**. By virtue of the present discussion, the revised statement should be further restated as follows: "Stereoisomeric relationship is subdivided into enantiomeric relationship and diastereomeric relationship, which are not exclusive."

In order to solve the question *how chirality and stereogenicity are distinct* in this paper, we consider that enantiomeric and diastereomeric relationships belong to distinct categories. The crux of the matter is, however, that *enantiomeric and diastereomeric relationships can occasionally overlap*.[6] Chirality is linked to enantiomeric relationship under point-group symmetry, while stereogenicity is linked to diastereomeric relationship under permutation group. In particular, we attach great importance to orbits, where enantiomers are considered to be the members of such an orbit as governed by point-group symmetry, while diastereomers are regarded as the members of an orbit under permutation-group symmetry.

2.2. Global/local stereogenicity versus prostereogenicity

As found in the expressions such as "As we have seen, 'stereogenicity' and 'local symmetry' are conceptually distinct" and "The crux of the matter is that chirotopicity and stereogenicity are conceptually distinct" in [10], the original term "stereogenicity" has been used in the meaning of the *local stereogenicity* that will be defined later in the present paper.[7] In other words, the counterpart concept of *global stereogenicity*

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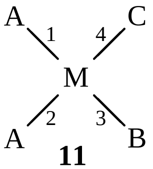
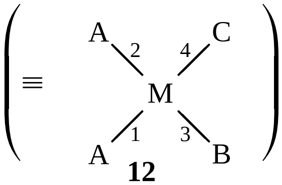
[5] For other approaches based on permutation groups, see [35–40].

[6] Our standpoint is essentially different from that of [1], which says "the dichotomy between enantiomers and diastereomers is unequivocal" on page 49. The overlap between enantiomeric and diastereomeric relationships affords a foundation of the validity of the CIP system.

[7] It should be emphasized that the terms *global/local* are closely linked to the concept of orbits that are governed by coset representations (CRs). In our approach, the pairwise consideration of the terms brings forth



11



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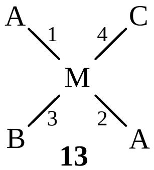

13

$4C'_{\mathrm{s}}(/C'_{\mathrm{s}})$
$\mathrm{A} + \mathrm{A} + \mathrm{B} + \mathrm{C}$
achiral

$C''_{2\mathrm{v}}(/C_{\mathrm{s}}) + 2C''_{2\mathrm{v}}(/C''_{2\mathrm{v}})$
$\mathrm{A}_2 + \mathrm{B} + \mathrm{C}$
achiral

$S_3^{[4]}(/S_1^{[4]}) + 2S_3^{[4]}(/S_3^{[4]})$
$\mathrm{A}_2 + \mathrm{B} + \mathrm{C}$
prostereogenic
(for $S_9^{[4]}$)

$S_3^{[4]}(/S_1^{[4]}) + 2S_3^{[4]}(/S_3^{[4]})$
$\mathrm{A}_2 + \mathrm{B} + \mathrm{C}$
stereogenic
(for $\pi S_9^{[4]}\pi^{-1}$)

Figure 3. Stereogenicity and prostereogenicity of square planar complexes with $\mathrm{A}_2\mathrm{BC}$.

has never been taken into consideration in the conventional terminology.[8] The missing of the latter concept is an unbalanced treatment in terms of the parallelism shown in table 1. The aim of this subsection is to clarify that the distinction between *global stereogenicity* and *local stereogenicity* is a crucial point to comprehend the nature of stereoisomerism.

Let us consider the stereogenicity of a square-planar complex with $\mathrm{A}_2\mathrm{BC}$ (figure 3). By exchanging the ligands A(2) and B(3), the complex **11** is converted into **13**, vice versa. Hence, **11** and **13** are decided to be diastereomeric. By virtue of the McCasland criterion, **11** and **13** are recognized as having “stereogenic centers”.

On the other hand, when one of the ligands A of **11** is replaced by a ligand X, we obtain a complex with AXBC, which is diastereomeric to another complex generated by the replacement of the other ligand A by X. This means that **11** involves a “prostereogenic center (element or unit)”. Since the same procedure applied to **13** gives no stereoisomers, the complex **13** is determined to have no “prostereogenic center (element or unit)”.

The discussion described in the preceding two paragraphs indicates that **11** is characterized to have a stereogenic center as well as a prostereogenic center in terms of the conventional terminology. This paradoxical result shows that such expressions as “stereogenic center” and “prostereogenic center” have the same drawback as expres-

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the concept of sphericity for point-group symmetry as well as the concept of tropicity for permutation-group symmetry.

[8] The usage of the same word “stereogenic” may cause confusion, since it is conceptually different from the original terminology of “stereogenicity”. We are able to say “stereogenic and astereogenic molecules” in terms of the new definition, just as we are able to say “chiral and achiral molecules”. Such expressions are not permissible under the original terminology. To avoid such confusion, we may coin a new term. But we adopt the same word, because the original “stereogenicity” can be rigorously characterized as *local stereogenicity* in terms of the new definition.

sions such as “chiral center” and “prochiral center” that have been recommended to be abandoned[9] [10].

Obviously, the procedure of determining the stereogenicity are conceptually different from that of determining the prostereogenicity. In fact, the stereogenicity of **11** or **13** is concerned with the diastereomeric relationship between **11** and **13**. From the present formulation (table 1), it implies the *global stereogenicity* of **11** (or **13**). On the other hand, the prostereogenicity of **11** (not of **13**) stems from its capability of generating the two diastereomers with AXBC. The present formulation indicates that it corresponds to the *local stereogenicity* of **11**, which brings the concept of tropicity, as shown below.

3. Relationships versus orbits as equivalence classes

In this section, we shall discuss enantiomeric relationship under point-group symmetry as well as diastereomeric relationship under permutation-group symmetry, so that we are able to reach a common standpoint to link the two relationships. Thereby, orbits under permutation-group symmetry are as important as orbits under point-group symmetry that have been extensively investigated [18]. For the sake of simplicity, we introduce several fundamental concepts by using allenes and ethylenes as examples of low symmetries.

3.1. *Enantiomeric relationship under point-group symmetry*

To comprehend point-group symmetry, let us examine molecules with A_2B_2 by starting from the skeletons listed in figure 1, where A and B are achiral ligands. The resulting molecules are depicted in figure 4. Among them, it is worthwhile to compare between the series of allenes (**14**, **15**, and **16**) and that of ethylenes (**17**, **18**, and **19**).

From the viewpoint of point-group symmetry, the allenes **14** and **15** are chiral and enantiomeric to each other, since they are mirror images that are interconvertible to each other with an appropriate rotoreflection. The operation of such a rotoreflection is regarded as an extended permutation $\overline{(1\;4)(2)(3)}$, which represents the conversion of such objects as A and B into their mirror images after the usual permutation (1 4)(2)(3). This extended permutation converts **14** into **15**. Note that a pair of enantiomers is conventionally defined as a pair of a stereoisomer and its mirror image that are not superimposable to each other. Strictly speaking, however, the enantiomer of **14** should be represented by the formula $\overline{\mathbf{14}}$, where $\overline{\mathrm{A}}$ and $\overline{\mathrm{B}}$ represent the enantiomeric ligands (mirror images) of A and B. The resulting $\overline{\mathbf{14}}$ is usually equalified to **15**, because $\overline{\mathrm{A}}$ and $\overline{\mathrm{B}}$ are respectively equal to A and B.[10]

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[9] This drawback stems from the word “center”. In terms of the present methodology, prostereogenicity does not come from a “center” but from an orbit of permutation-group type. This is an analogy with the fact that prochirality does not come from a “center” but from an orbit of point-group type [16].

[10] Note that, if A (or B) has an inner structure that is susceptible of environmental disturbance, the equality between the inner structure of the A and that of the $\overline{\mathrm{A}}$ (or between the counterparts of B and $\overline{\mathrm{B}}$) is

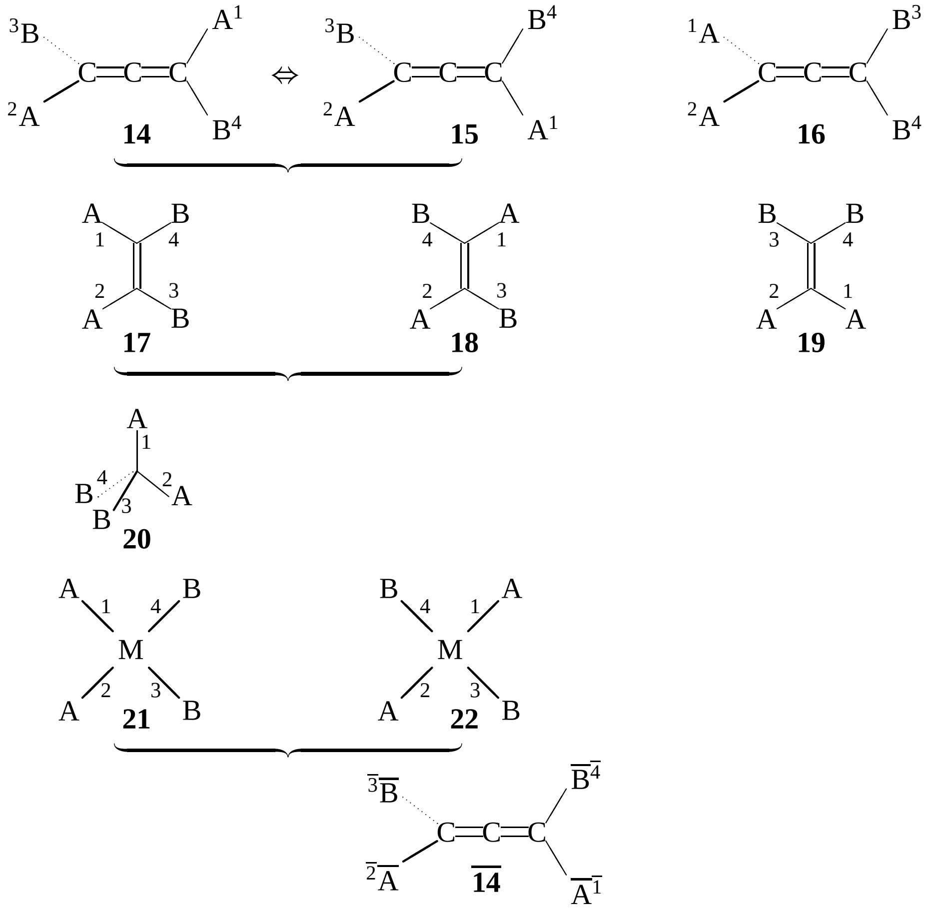

Figure 4. Molecules of A_2B_2. A brace is used to designate a diastereomeric relationship. An arrow (⇔) designates an enantiomeric relationship.

On the other hand, the ethylene **17** is converted into itself by a rotoreflection (e.g., $\overline{(1)(2)(3)(4)}$), while the ethylene **18** is distinctly converted into itself with the rotoreflection. The extended permutation $\overline{(1)(2)(3)(4)}$ generates ethylenes having $\overline{A}_2\overline{B}_2$ in place of A_2B_2. They are identical with **17** and **18**, since $\overline{A}$ and $\overline{B}$ are respectively equal A

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not always obvious. Stereochemistry should properly deal with a set of ligands or functional groups that have their own inner structures. In contrast, geometry essentially deals with an assembly of points that have no inner structures. This difference between stereochemistry and geometry cannot be too strongly emphasized, because the conventional stereochemistry has implicitly regarded a ligand as being structureless, i.e., as a point in the same way as geometry. In this context, stereoisomers cannot be recognized as prototypes of permutational isomers [41], if we take account of ligands having inner structures. Note that enantiomers having chiral ligands cannot be formulated as permutational isomers. Our standpoint is that stereoisomers are considered to be dually and distinctly controlled by point-groups and permutation groups.

and B. The ethylenes **17** and **18** are concluded to be achiral and diastereomeric to each other, because diastereomers are conventionally defined as stereoisomers that are not enantiomers.

3.2. Diastereomeric relationship under permutation-group symmetry

From a permutational point of view, the allene **14** is converted into its enantiomer **15** by a permutation between A(1) and B(4), i.e., (1 4)(2)(3). As discussed in the preceding subsection, the enantiomer of **14** is represented by the formula $\overline{\mathbf{14}}$, although the latter is equalified to **15**. This means that, from a permutational point of view, the relationship between **14** and **15** is recognized as being diastereomeric rather than as being enantiomeric.

This conclusion is reinforced by examining the ethylenes **17** and **18**. Thus, the ethylene **17** is converted into **18** by the same permutation (1 4)(2)(3), where **17** and **18** are achiral and diasteromeric to each other. If we take account of such permutability and if we do not consider rotoreflection operators, the relationship between **17** and **18** is determined to be the same type as that between **14** and **15**.

The data for derivatives of $\mathrm{A}_2\mathrm{p}_2$ (and $\mathrm{A}_2\overline{\mathrm{p}}_2$) collected in figure 5 give more pieces of information on their permutability. The permutation (1 4)(2)(3) converts a chiral allene **23** into the corresponding diastereomer **24**. In this process and related permutations, the corresponding enantiomeric allenes $\overline{\mathbf{23}}$ and $\overline{\mathbf{24}}$ do not appear, since a chiral ligand p cannot be converted into its enantiomeric ligand $\overline{\mathrm{p}}$ by any usual permutations because of no rotoreflections. Note that the parallelism between the pair of **14** and $\overline{\mathbf{14}}$ in figure 4 and the pair of **23** and $\overline{\mathbf{23}}$ in figure 5. It follows that the usual permutations are incapable of treating enantiomeric relationships between chiral molecules.

3.3. Orbits for point groups and for permutation groups

From the stereochemical point of view, the four positions of each skeleton (figure 1) construct an orbit (equivalence class), which is governed by a coset representation (CR). As a result, we obtain $D_{2\mathrm{d}}(/C_{\mathrm{s}})$ for the allene skeleton (**1**), $D_{2\mathrm{h}}(/C_{\mathrm{s}}'')$ for the ethylene skeleton (**2**), a CR $T_{\mathrm{d}}(/C_{3\mathrm{v}})$ for the tetrahedral skeleton (**3**), and $D_{4\mathrm{h}}(/C_{2\mathrm{v}}'')$ for the square-planar skeleton (**4**).

The point-group symmetry of a molecule derived from a given skeleton is determined under the action of the point group of the skeleton. For example, the point-group symmetry of the allene **23** is determined under the point group $D_{2\mathrm{d}}$, as shown in figure 6, where an end-on view from right is adopted as a convention for drawing allene derivatives. The four positions are permuted by the permutations of the CR $D_{2\mathrm{d}}(/C_{\mathrm{s}})$, where positions 1 and 2 accommodate an achiral ligand A and positions 3 and 4 accommodate chiral ligands p and $\overline{\mathrm{p}}$. Among the resulting eight molecules (**23a** to **23d**; and $\overline{\mathbf{23a}}$ to $\overline{\mathbf{23d}}$), the two molecules **23a** and **23b** remain invariant. The permutations that keep the derivative **23** invariant are found to correspond to the symmetry operations I and $C_{2(1)}$, which construct a subgroup C_2'. Thereby, the derivative **23** is determined to belong to the C_2'-symmetry, which is a subsymmetry of $D_{2\mathrm{d}}$.

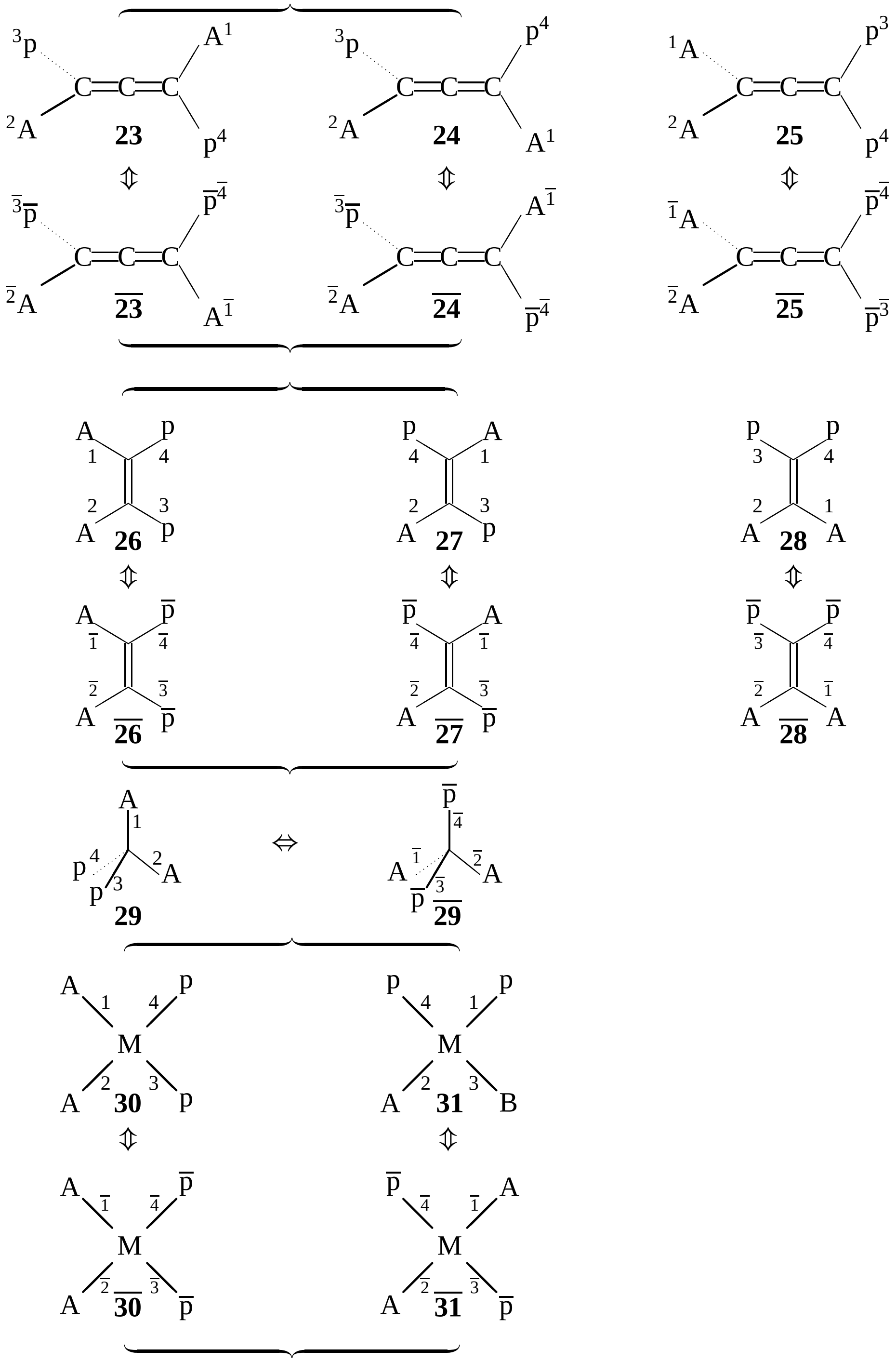

Figure 5. Molecules of A_2p_2 and $A_2\overline{p}_2$. See the caption of figure 4.

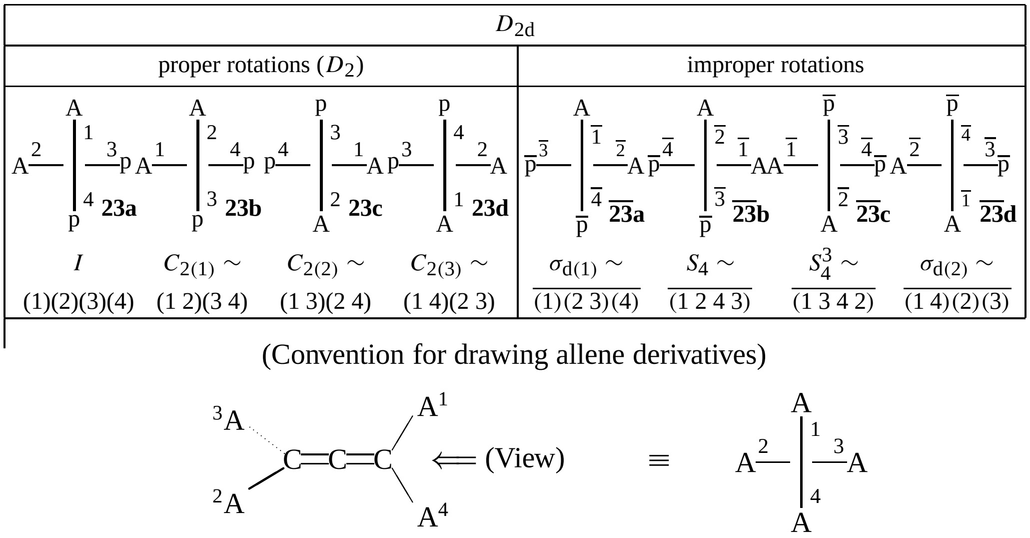

Figure 6. Isomer equivalence for an allene derivative of C'_2-symmetry with A^2p^2 (and $A^2\overline{p}^2$) under the point group D_{2d}.

According to the action of D_{2d}, the set of the eight molecules are divided into two sets of homomers ({**23a**, **23b**} and {**23c**, **23d**}) and two sets of enantiomers ({$\overline{\mathbf{23a}}$, $\overline{\mathbf{23b}}$} and {$\overline{\mathbf{23c}}$, $\overline{\mathbf{23d}}$}). These four sets construct an orbit governed by the CR $D_{2d}(/C'_2)$, where the local symmetry (C'_2) of the CR corresponds to the symmetry of **23**.[11] The $D_{2d}(/C'_2)$-orbit is enantiospheric so that it accommodates the four sets in agreement with a compensated chiral packing for the enantiosphericity of the $D_{2d}(/C'_2)$-orbit [18,21]. Note that the size of the $D_{2d}(/C'_2)$-orbit is calculated to be $|D_{2d}|/|C'_2| = 8/2 = 4$, where the symbols $|D_{2d}|$ and $|C'_2|$ represent the orders of the respective groups.

The permutation-group symmetry of a molecule derived from a given skeleton is determined under the action of the permutation group of the skeleton. For example, the permutation-group symmetry of the allene **23** is determined under the permutation group $S_9^{[4]}$, as shown in figure 7. The permutation group $S_9^{[4]}$ is a subgroup of the symmetric group of degree 4 ($S^{[4]}$).[12] The resulting four molecules (**23a** to **23d**) are homomeric and the remaining four (**24a** to **24d**) are diastereomeric to **23**, where the top two molecules **23a** and **23b** remain invariant. Hence, the permutation group $S_{2'}^{[4]}$ ($= \{(1)(2)(3)(4), (1\ 2)(3\ 4)\}$) is ascribed to the derivative **23**.

According to the action of $S_9^{[4]}$ (figure 7), the set of the eight molecules are divided into two sets of homomers ({**23a**, **23b**} and {**23c**, **23d**}) and two sets of diastereomers ({**24a**, **24b**} and {**24c**, **24d**}), which construct an orbit governed by the CR $S^{[4]}(/S_{2'}^{[4]})$.

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[11] Such orbit formation has been used to formulate new tools of combinatorial enumeration. See [18, chapters 15 and 20]. A more elaborate discussion for tetrahedral molecules has been reported [30].

[12] We use the symbol $S^{[4]}$ for denoting the symmetric group of degree 4 in order to differentiate it from the point group S_4.

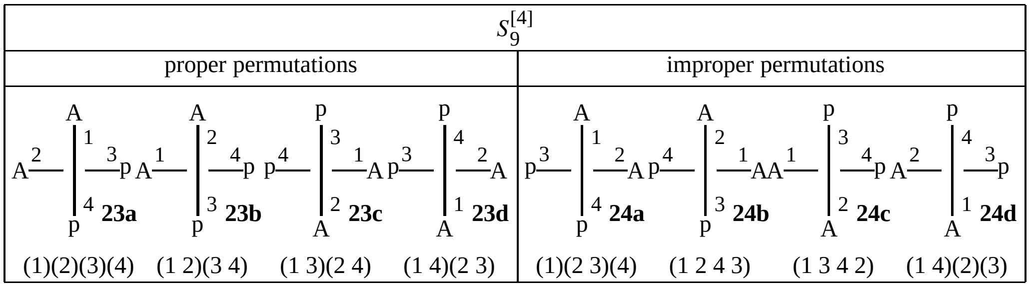

Figure 7. Isomer equivalence of an allene derivative of C_2'-symmetry with A^2p^2 under the permutation group $S_9^{[4]}$.

Note that the size of the $S^{[4]}(/S_{2'}^{[4]})$-orbit is calculated to be $|S^{[4]}|/|S_{2'}^{[4]}| = 8/2 = 4$, where the symbols $|S^{[4]}|$ and $|S_{2'}^{[4]}|$ represent the orders of the respective groups. The local symmetry ($S_{2'}^{[4]}$) of the CR corresponds to the permutation-group symmetry of the derivative **23**.

It should be emphasized here that the allene **23** is dually and distinctly controlled by a CR of point-group type $D_{2d}(/C_2')$ as well as by a CR of permutational type $S^{[4]}(/S_{2'}^{[4]})$. On the same line as we have considered global/local symmetries for CRs of the former type [18], it is natural to consider global/local symmetries for CRs of the latter type. This point will be later discussed in detail for the proposal of the concept of tropicity. Moreover, the conversion under the point group D_{2d} (figure 6) produces a pair of enantiomers (**23** and $\overline{\mathbf{23}}$), while the conversion under the permutation group $S^{[4]}$ (figure 7) generates a duplex of diastereomers (**23** and **24**).

3.4. *Stereoisomeric groups*

The point-group symmetry shown in figure 6 indicates the enantiomeric relationship between **23** and $\overline{\mathbf{23}}$ (or between **24** and $\overline{\mathbf{24}}$). On the other hand, the permutation-group symmetry shown in figure 7 reveals the diastereomeric relationship between **23** and **24** (or between $\overline{\mathbf{23}}$ and $\overline{\mathbf{24}}$). However, another diastereomeric relationship between **23** and $\overline{\mathbf{24}}$ (or between **24** and $\overline{\mathbf{23}}$) cannot be decided. To accomplish this decision for the allene skeleton **1**, we should combine the point-group symmetry with permutation-group symmetry, as shown in figure 8.[13] The resulting group is represented by the direct product $S_9^{[4]} \times S_\sigma$, where we place $S_\sigma = \{(1)(2)(3)(4), \overline{(1)(2)(3)(4)}\}$. It is called *a stereoisomeric group*. Thereby, the permutation group $S_9^{[4]}$ is a subgroup of the stereoisomeric group. When $S_7^{[4]}$ ($\subset S_9^{[4]}$) is selected to be isomorphic to D_2 (the group of the proper

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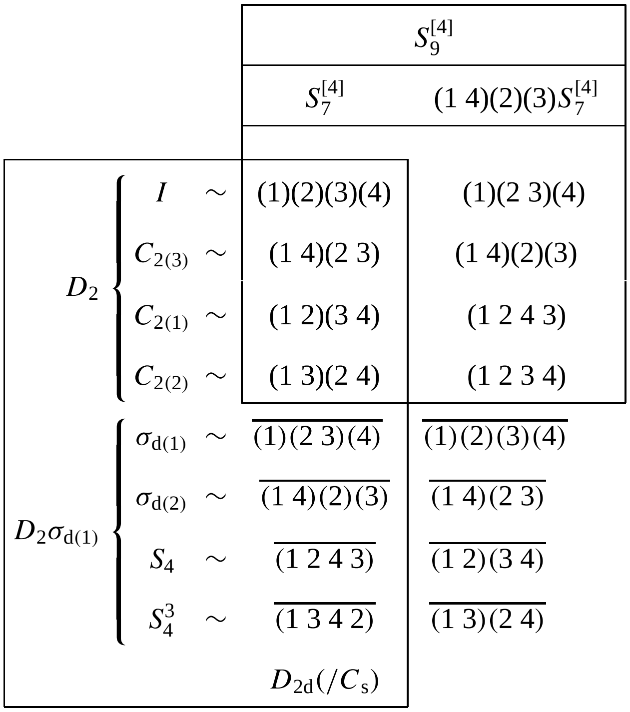
[13] More general groups are hyperoctahedral groups described in in [42, chapter II], where equivalent counterparts to the present stereoisomeric groups have been defined as subgroups of such hyperoctahedral groups. They have been treated in terms of linear representations [42], while our treatment is based on permutation representations and coset representations.



		$S_9^{[4]}$	
		$S_7^{[4]}$	$(1\,4)(2)(3)S_7^{[4]}$
D_2	$I \sim$	$(1)(2)(3)(4)$	$(1)(2\,3)(4)$
	$C_{2(3)} \sim$	$(1\,4)(2\,3)$	$(1\,4)(2)(3)$
	$C_{2(1)} \sim$	$(1\,2)(3\,4)$	$(1\,2\,4\,3)$
	$C_{2(2)} \sim$	$(1\,3)(2\,4)$	$(1\,2\,3\,4)$
$D_2\sigma_{d(1)}$	$\sigma_{d(1)} \sim$	$\overline{(1)(2\,3)(4)}$	$\overline{(1)(2)(3)(4)}$
	$\sigma_{d(2)} \sim$	$\overline{(1\,4)(2)(3)}$	$\overline{(1\,4)(2\,3)}$
	$S_4 \sim$	$\overline{(1\,2\,4\,3)}$	$\overline{(1\,2)(3\,4)}$
	$S_4^3 \sim$	$\overline{(1\,3\,4\,2)}$	$\overline{(1\,3)(2\,4)}$
		$D_{2d}(/C_s)$	

Figure 8. $D_{2d}(/C_s)$ and $S_9^{[4]}$.

rotations of D_{2d}), the point-group symmetry D_{2d} corresponds to the following extended permutation group:

$$S_7^{[4]} + \overline{(1)(2\,3)(4)}S_7^{[4]}, \tag{1}$$

which can be equalified to the CR $D_{2d}(/C_s)$ (figure 8).

As for the ethylene skeleton **2**, the enantiomeric relationship between **26** and $\overline{\mathbf{26}}$ (or between **27** and $\overline{\mathbf{27}}$) as well as the diastereomeric relationships between **26** and **27** (or between $\overline{\mathbf{26}}$ and $\overline{\mathbf{27}}$) and between **26** and $\overline{\mathbf{27}}$ (or between **27** and $\overline{\mathbf{26}}$) can be treated by the direct product $S_9^{[4]} \times S_\sigma$, which is a stereoisomeric group for the ethylene skeleton, as shown in figure 9. The point-group symmetry D_{2h} corresponds to $S_7^{[4]} \times S_\sigma$, which can be equalified to the CR $D_{2h}(/C_s'')$ (figure 9).

The four positions of the tetrahedral skeleton **3** construct an orbit of point-group symmetry, which is governed by the CR $T_d(/C_{3v})$. The permutations for the maximal chiral group T are equalified to those involved in the isomorphic subgroup $S_{10}^{[4]}$ of the symmetric group of degree 4 ($S^{[4]}$). Thereby, we obtain the corresponding stereoisomeric group represented by $S^{[4]} \times S_\sigma$, as shown in figure 10. The point-group symmetry T_d corresponds to the following set:

$$S_{10}^{[4]} + \overline{(1)(2\,3)(4)}S_{10}^{[4]}, \tag{2}$$

			$S_9^{[4]}$	
			$S_7^{[4]}$	$(1\ 4)(2)(3)S_7^{[4]}$
D_2	I	$\sim$	$(1)(2)(3)(4)$	$(1\ 4)(2)(3)$
	$C_{2(3)}$	$\sim$	$(1\ 3)(2\ 4)$	$(1\ 3\ 4\ 2)$
	$C_{2(1)}$	$\sim$	$(1\ 2)(3\ 4)$	$(1\ 2\ 3\ 4)$
	$C_{2(2)}$	$\sim$	$(1\ 4)(2\ 3)$	$(1)(2\ 3)(4)$
$D_2\sigma_\mathrm{h}$	σ_h	$\sim$	$\overline{(1)(2)(3)(4)}$	$\overline{(1\ 4)(2)(3)}$
	i	$\sim$	$\overline{(1\ 3)(2\ 4)}$	$\overline{(1\ 3\ 4\ 2)}$
	$\sigma_{\mathrm{d}(1)}$	$\sim$	$\overline{(1\ 2)(3\ 4)}$	$\overline{(1\ 2\ 3\ 4)}$
	$\sigma_{\mathrm{d}(2)}$	$\sim$	$\overline{(1\ 4)(2\ 3)}$	$\overline{(1)(2\ 3)(4)}$
			$D_{2\mathrm{h}}(/C''_\mathrm{s})$	

Figure 9. $D_{2\mathrm{h}}(/C''_\mathrm{s})$ and $S_9^{[4]}$.

which can be equalified to the CR $T_\mathrm{d}(/C_{3\mathrm{v}})$ (figure 10). On the other hand, the symmetric group of degree 4 ($S^{[4]}$) corresponds to the following set:

$$S^{[4]} = S_{10}^{[4]} + (1)(2\ 3)(4)S_{10}^{[4]}. \tag{3}$$

The four positions of the square-planar skeleton **4** construct an orbit governed by the CR $D_{4\mathrm{h}}(/C''_{2\mathrm{v}})$. The permutations for the maximal chiral group is D_4 are equalified to those involved in the isomorphic subgroup $S_9^{[4]}$ of the symmetric group of degree 4 ($S^{[4]}$). Thereby, we obtain the corresponding stereoisomeric group represented by $S^{[4]} \times S_\sigma$, as shown in figure 11. The point-group symmetry $D_{4\mathrm{h}}$ corresponds to the direct product $S_9^{[4]} \times S_\sigma$, which can be equalified to the CR $D_{4\mathrm{h}}(/C''_{2\mathrm{v}})$ (figure 11). On the other hand, the symmetric group of degree 4 ($S^{[4]}$) corresponds to the following set:

$$S^{[4]} = S_9^{[4]} + (1)(2\ 4\ 3)S_9^{[4]} + (1)(2\ 3\ 4)S_9^{[4]}. \tag{4}$$

The stereoisomeric group for the allene skeleton (figure 8) is the same as that for the ethylene skeleton (figure 9). On the same line, the stereoisomeric group for the tetrahedral skeleton (figure 10) is the same as that for the square-planar skeleton (figure 11). The difference between the skeletons at issue causes, however, the appearance of various modes of stereoisomerism.

			$S^{[4]}$	
			$S_{10}^{[4]}$	$(1)(2\ 3)(4)S_{10}^{[4]}$
T	I	$\sim$	$(1)(2)(3)(4)$	$(1)(2\ 3)(4)$
	$C_{2(1)}$	$\sim$	$(1\ 2)(3\ 4)$	$(1\ 2\ 4\ 3)$
	$C_{2(2)}$	$\sim$	$(1\ 3)(2\ 4)$	$(1\ 3\ 4\ 2)$
	$C_{2(3)}$	$\sim$	$(1\ 4)(2\ 3)$	$(1\ 4)(2)(3)$
	$C_{3(1)}$	$\sim$	$(1)(2\ 4\ 3)$	$(1)(2)(3\ 4)$
	$C_{3(3)}$	$\sim$	$(1\ 2\ 3)(4)$	$(1\ 2)(3)(4)$
	$C_{3(2)}$	$\sim$	$(1\ 3\ 4)(2)$	$(1\ 3\ 2\ 4)$
	$C_{3(4)}$	$\sim$	$(1\ 4\ 2)(3)$	$(1\ 4\ 2\ 3)$
	$C_{3(1)}^2$	$\sim$	$(1)(2\ 3\ 4)$	$(1)(2\ 4)(3)$
	$C_{3(4)}^2$	$\sim$	$(1\ 2\ 4)(3)$	$(1\ 2\ 3\ 4)$
	$C_{3(3)}^2$	$\sim$	$(1\ 3\ 2)(4)$	$(1\ 3)(2)(4)$
	$C_{3(2)}^2$	$\sim$	$(1\ 4\ 3)(2)$	$(1\ 4\ 3\ 2)$
$T\sigma_{d(1)}$	$\sigma_{d(1)}$	$\sim$	$\overline{(1)(2\ 3)(4)}$	$\overline{(1)(2)(3)(4)}$
	$S_{4(3)}$	$\sim$	$\overline{(1\ 2\ 4\ 3)}$	$\overline{(1\ 2)(3\ 4)}$
	$S_{4(3)}^3$	$\sim$	$\overline{(1\ 3\ 4\ 2)}$	$\overline{(1\ 3)(2\ 4)}$
	$\sigma_{d(6)}$	$\sim$	$\overline{(1\ 4)(2)(3)}$	$\overline{(1\ 4)(2\ 3)}$
	$\sigma_{d(2)}$	$\sim$	$\overline{(1)(2)(3\ 4)}$	$\overline{(1)(2\ 4\ 3)}$
	$\sigma_{d(4)}$	$\sim$	$\overline{(1\ 2)(3)(4)}$	$\overline{(1\ 2\ 3)(4)}$
	$S_{4(1)}$	$\sim$	$\overline{(1\ 3\ 2\ 4)}$	$\overline{(1\ 3\ 4)(2)}$
	$S_{4(1)}^3$	$\sim$	$\overline{(1\ 4\ 2\ 3)}$	$\overline{(1\ 4\ 2)(3)}$
	$\sigma_{d(3)}$	$\sim$	$\overline{(1)(2\ 4)(3)}$	$\overline{(1)(2\ 3\ 4)}$
	$S_{4(2)}^3$	$\sim$	$\overline{(1\ 2\ 3\ 4)}$	$\overline{(1\ 2\ 4)(3)}$
	$\sigma_{d(5)}$	$\sim$	$\overline{(1\ 3)(2)(4)}$	$\overline{(1\ 3\ 2)(4)}$
	$S_{4(2)}$	$\sim$	$\overline{(1\ 4\ 3\ 2)}$	$\overline{(1\ 4\ 3)(2)}$
			$T_d(/C_{3v})$	

Figure 10. $T_d(/C_{3v})$ and $S^{[4]}$.

			$S^{[4]}$		
			$S_9^{[4]}$	$(1)(2\,4\,3)S_9^{[4]}$	$(1)(2\,3\,4)S_9^{[4]}$
D_4	I	$\sim$	$(1)(2)(3)(4)$	$(1)(2\,4\,3)$	$(1)(2\,3\,4)$
	C_4	$\sim$	$(1\,2\,3\,4)$	$(1\,4)(2)(3)$	$(1\,3\,2\,4)$
	$C_{2(3)}$	$\sim$	$(1\,3)(2\,4)$	$(1\,2\,3)(4)$	$(1\,4\,3)(2)$
	C_4^3	$\sim$	$(1\,4\,3\,2)$	$(1\,3\,4\,2)$	$(1\,2)(3)(4)$
	$C_{2(1)}$	$\sim$	$(1)(2\,4)(3)$	$(1)(2\,3)(4)$	$(1)(2)(3\,4)$
	$C'_{2(1)}$	$\sim$	$(1\,2)(3\,4)$	$(1\,4\,2)(3)$	$(1\,3\,2)(4)$
	$C_{2(2)}$	$\sim$	$(1\,3)(2)(4)$	$(1\,2\,4\,3)$	$(1\,4\,2\,3)$
	$C'_{2(2)}$	$\sim$	$(1\,4)(2\,3)$	$(1\,3\,4)(2)$	$(1\,2\,4)(3)$
$D_4\sigma_h$	σ_h	$\sim$	$\overline{(1)(2)(3)(4)}$	$\overline{(1)(2\,4\,3)}$	$\overline{(1)(2\,3\,4)}$
	S_4	$\sim$	$\overline{(1\,2\,3\,4)}$	$\overline{(1\,4)(2)(3)}$	$\overline{(1\,3\,2\,4)}$
	i	$\sim$	$\overline{(1\,3)(2\,4)}$	$\overline{(1\,2\,3)(4)}$	$\overline{(1\,4\,3)(2)}$
	S_4^3	$\sim$	$\overline{(1\,4\,3\,2)}$	$\overline{(1\,3\,4\,2)}$	$\overline{(1\,2)(3)(4)}$
	$\sigma_{v(1)}$	$\sim$	$\overline{(1)(2\,4)(3)}$	$\overline{(1)(2\,3)(4)}$	$\overline{(1)(2)(3\,4)}$
	$\sigma_{d(1)}$	$\sim$	$\overline{(1\,2)(3\,4)}$	$\overline{(1\,4\,2)(3)}$	$\overline{(1\,3\,2)(4)}$
	$\sigma_{v(2)}$	$\sim$	$\overline{(1\,3)(2)(4)}$	$\overline{(1\,2\,4\,3)}$	$\overline{(1\,4\,2\,3)}$
	$\sigma_{d(2)}$	$\sim$	$\overline{(1\,4)(2\,3)}$	$\overline{(1\,3\,4)(2)}$	$\overline{(1\,2\,4)(3)}$
			$D_{4h}(/C''_{2v})$		

Figure 11. $D_{4h}(/C''_{2v})$ and $S^{[4]}$.

4. Stereogenicity of permutation groups

4.1. Proper and improper permutations

For the sake of simplicity in discussing allene and ethylene derivatives, we use the permutation group $S_9^{[4]}$ in place of the corresponding stereoisomeric group such as $S_9^{[4]} \times S_\sigma$. On the same line, in order to discuss tetrahedral and square-planar derivatives, we use the symmetric group $S^{[4]}$ in place of the corresponding stereoisomeric group. This means that a pair of enantiomers is regarded as an object of $S_9^{[4]}$ in case of allene or ethylene derivatives and as an object of $S^{[4]}$ in case of tetrahedral or square-planar derivatives.

For example, the pair of **23** and $\overline{\mathbf{23}}$ is diastereomeric to the other pair of **24** and $\overline{\mathbf{24}}$ under the permutation group $S_9^{[4]}$. In this context, the pair of **14** and **15** is considered to be an object behaving as a self-diastereomer under $S_9^{[4]}$.[14]

The permutations contained in $S_9^{[4]}$ are here classified into proper permutations and improper permutations.[15] A *proper permutation* is defined as being equal to a permutation contained in the CR that correspond to the maximal chiral subgroup (D_2) of $D_{2\mathrm{d}}$ (or $D_{2\mathrm{h}}$). They construct a permutation group $S_7^{[4]}$, which is isomorphic to D_2. Permutations other than proper rotations are defined as *improper permutations*. Such improper permutations are involved in the coset $(1)(2\ 3)(4)S_7^{[4]}$ found in figure 8 or in the coset $(1\ 4)(2)(3)S_7^{[4]}$ found in figure 9. Note that the same permutation group $S_9^{[4]}$ can be applied to allenes and ethylenes, as found in figures 8 and 9.

As for the tetrahedral skeleton **3**, the permutations contained in $S^{[4]}$ are also classified into proper permutations and improper permutations (figure 10). Thus, the twelve permutations contained in $S_{10}^{[4]}$ are proper permutations.[16] The remaining twelve permutations contained in the coset $(1)(2\ 3)(4)S_{10}^{[4]}$ are improper rotations.

On the other hand, the square-planar skeleton **4** requires an alternative classification of the permutations contained in $S^{[4]}$. According to figure 11, the eight permutations contained in $S_9^{[4]}$ are proper permutations. The remaining sixteen permutations contained in the cosets $(1)(2\ 4\ 3)S_9^{[4]}$ and $(1)(2\ 3\ 4)S_9^{[4]}$ are improper permutations.[17]

4.2. *Stereogenic and astereogenic groups*

In terms of proper and improper permutations, the subgroups of the permutation groups $S_9^{[4]}$ and $S^{[4]}$ are classified into stereogenic and astereogenic groups. A *stereogenic* group is defined as a group that consists of proper permutations only, while an *astereogenic* group is defined as a group that has at least one improper permutation.[18] Thereby, *the maximal stereogenic subgroup* of a given astereogenic group is

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[14] If a pair of enantiomers is converted into itself under permutation symmetry, it is defined as being self-diastereomeric.

[15] Do not confuse proper and improper permutations with odd and even permutations.

[16] Note that $S_{10}^{[4]}$ is isomorphic to the point group T.

[17] The group $S_9^{[4]}$ is conjugated to $\pi S_9^{[4]}\pi^{-1}$, where π is selected to be $(1)(2\ 4\ 3)$ or $(1)(2\ 3\ 4)$.

[18] If a molecule coincides with itself by means of proper permutations only, it is defined as being stereogenic; otherwise, it is defined as being astereogenic. A stereogenic molecule belongs to a stereogenic permutation group, while an astereogenic molecule belongs to an astereogenic permutation group. Note that this definition is parallel to that of chiral and achiral molecules. Thus, a chiral molecule coincides with itself under the action of proper rotations so as to belong to a chiral point group, while an achiral molecule coincides with itself under the action of proper and improper rotations so as to belong to an achiral point group.

Table 2
Stereogenic and astereogenic groups for ligancy 4[a,b].

Permutation group	Allene	Ethylene	Tetrahedron	Square-planar	Remark[c,d]
$S^{[4]}$	–	–	a-stereogenic	a-stereogenic	$\sim T_{\mathrm{d}}$
$S_{10}^{[4]}$	–	–	* stereogenic	a-stereogenic	$\sim T$
$S_9^{[4]}$	a-stereogenic	a-stereogenic	a-stereogenic	* stereogenic	$\sim D_{2\mathrm{d}}$
$S_8^{[4]}$	–	–	a-stereogenic	a-stereogenic	$\sim C_{3\mathrm{v}}$
$S_7^{[4]}$	a-stereogenic	a-stereogenic	a-stereogenic	stereogenic a-stereogenic	$\sim C_{2\mathrm{v}}$
$S_6^{[4]}$	*stereogenic	*stereogenic	stereogenic	stereogenic	$\sim D_2$
$S_5^{[4]}$	a-stereogenic	a-stereogenic	a-stereogenic	stereogenic	$\sim S_4$
$S_4^{[4]}$	–	–	stereogenic	a-stereogenic	$\sim C_3$
$S_3^{[4]}$	a-stereogenic	a-stereogenic	a-stereogenic	stereogenic a-stereogenic	$\sim C_{\mathrm{s}}$
$S_{2'}^{[4]}$	stereogenic	stereogenic	–[e]	–[e]	$\sim C'_2$
$S_2^{[4]}$	stereogenic	stereogenic	stereogenic	stereogenic	$\sim C_2$
$S_1^{[4]}$	stereogenic	stereogenic	stereogenic	stereogenic	$\sim C_1$

[a] The word "astereogenic" is spelled as "a-stereogenic" for the sake of visibility.
[b] An asterisked group is the maximal stereogenic subgroup.
[c] The symmetric group of degree 4 ($S^{[4]}$) is isomorphic to the point group T_{d}.
[d] Do not confuse the point group (S_4) with the symmetric group of degree 4 ($S^{[4]}$).
[e] Omitted because of conjugacy.

defined as a stereogenic subgroup that contains all of the proper permutations of the astereogenic group. The maximal stereogenic subgroup can accompany several conjugate subgroups. For example, when we consider the permutation group $S_9^{[4]}$, astereogenic subgroups are found to be $S_9^{[4]}$, $S_7^{[4]}$, $S_5^{[4]}$, and $S_3^{[4]}$, while stereogenic subgroups are $S_6^{[4]}$, $S_{2'}^{[4]}$, $S_2^{[4]}$, and $S_1^{[4]}$. The data are summarized in table 2, along with the data for $S^{[4]}$.

In general, a stereogenic subgroup is a subgroup of the maximal stereogenic subgroup. However, an astereogenic subgroup and a stereogenic subgroup can be occasionally conjugate to each other, as found for the subgroups $S_7^{[4]}$ and $S_3^{[4]}$. These cases will be discussed below in detail in the discussion on square-planar cases.

The stereogenicity on the basis of this classification is useful to examine the existence of diastereoisomerism. For example, the allene **14** belongs to $S_{2'}^{[4]}$, which is stereogenic. Hence, there appears the corresponding diastereomer **15**. On the same line, the allene **23** (or $\overline{\mathbf{23}}$) belongs to $S_{2'}^{[4]}$, which is stereogenic so as to give the corresponding diastereomer **24** (or $\overline{\mathbf{24}}$). The achiral ethylene **17** and the chiral ethylene **26** both belong to $S_{2'}^{[4]}$. The stereogenicity of $S_{2'}^{[4]}$ explains the existence of diastereomers, as shown in figures 4 and 5.

The astereogenicity is useful to test the non-existence of diastereoisomerism. For example, the allene **16** belongs to $S_7^{[4]}$, which is astereogenic. Hence, there appears no diastereomer. The chiral allene **25** also belongs to $S_7^{[4]}$, which is astereogenic. The achiral ethylene **19** and the chiral ethylene **28** belong to $S_7^{[4]}$, which is astereogenic so as to generate no diastereomers.

5. The concept of tropicity

5.1. *Orbits under permutation-group symmetry*

In the above discussions, the four positions of each skeleton of figure 1 are regarded to be controlled by an appropriate permutation group such as $S_9^{[4]}$ (for allene and ethylene) and $S^{[4]}$ (for tetrahedral and square-planar molecules). A more elaborate treatment is necessary to investigate the modes of ligand substitution. On the same line as a CR $G(/G_i)$ of a point group, we can consider a CR $P(/P_i)$ of a permutation group. For example, the four positions in the allene skeleton (**1**) construct an orbit of permutational type, which is governed by a CR represented by the symbol $S_9^{[4]}(/S_3^{[4]})$, as shown in figure 1. The size of the orbit is calculated to be $|S_9^{[4]}|/|S_3^{[4]}| = 8/2 = 4$. Obviously, the CR $S_9^{[4]}(/S_3^{[4]})$ can be equalified to the permutation group $S_9^{[4]}$. In a parallel fashion to the stereochemical consideration (the left column of table 1), we introduce two parallel concepts for the CR, i.e., the global permutation-group symmetry $S_9^{[4]}$ and the local permutation-group symmetry $S_3^{[4]}$, where we place

$$S_3^{[4]} = \big\{(1)(2)(3)(4), (1)(2\ 3)(4)\big\}. \tag{5}$$

Note that position 1 is fixed by the permutations of $S_3^{[4]}$ under the permutation symmetry $S_9^{[4]}$, as found in figure 8. The CRs for the other skeletons are also collected in figure 1.

5.2. *Definition of tropicity*

Stereogenicity/astereogenicity introduced in the previous section is found to be parallel to chirality/achirality. In addition, a CR $P(/P_i)$ of a permutation group can be characterize on the same line as a CR $G(/G_i)$ of a point group. This parallelism creates important concepts listed in table 1.

5.2.1. *Global and local stereogenicity/astereogenicity*

By starting from the concept of chirality/achirality, we have proposed the concept of sphericity based on a CR $G(/G_i)$ that is ascribed to an orbit (equivalence class) of objects [16]. The sphericity is determined by examining whether the global symmetry G and the local symmetry G_i are chiral or achiral, as shown in table 3. Each orbit accommodates a set of ligands in terms of chirality fittingness that is controlled by the sphericity of the orbit. Thereby, the concept of prochirality is introduced as an attribute

Table 3
Sphericity of $G(/G_i)$ [16].

Global (point-group) symmetry G	Local (point-group) symmetry G_i	Sphericity of $G(/G_i)$	Chirality fittingness (objects allowed)
achiral	achiral	homospheric	achiral
achiral	chiral	enantiospheric	achiral[a], chiral[b]
chiral	chiral	hemispheric	achiral[a], chiral

[a] An achiral object in this orbit is desymmetrized into a chiral one.
[b] A compensated chiral packing of enantiomeric objects of opposite chiralities.

Table 4
Tropicity of $P(/P_i)$ [16].

Global (permutation-group) symmetry P	Local (permutation-group) symmetry P_i	Tropicity of $P(/P_i)$	Stereogenicity fittingness[a]
astereogenic	astereogenic	homotropic	achiral, chiral
astereogenic	stereogenic	enantiotropic	achiral[b,c], chiral[b,c]
stereogenic	stereogenic	hemitropic	achiral[b], chiral[b]

[a] Strictly speaking, an astereogenic or stereogenic object should be considered. However, this consideration is too complex to give no straight-forward discussion.
[b] An achiral or chiral object is restricted to be stereogenic.
[c] The mode of accommodation is characterized as a compensated stereogenic packing on the analogy of a compensated chiral packing. When we take account of an enantiotropic orbit of stereoisomerism, the orbit accommodates diastereomeric objects of opposite stereogenicities (see figure 7).

of an enantiospheric orbit: *a molecule is defined as being prochiral if it has at least one enantiospheric orbit* [16].

We here focus our attention on the parallelism between the concept of stereogenicity/astereogenicity and that of chirality/achirality. More specifically, we take account of the parallelism between the global/local permutation-group symmetries and the global/local point-group symmetries. Thereby, we propose the concept of tropicity[19] (table 4) on the analogy of the concept of sphericity (table 3). As a result, the concept of prostereogenicity is introduced as an attribute of an enantiotropic orbit: *a molecule is defined as being prostereogenic if it has at least one enantiotropic orbit.* Obviously, this is parallel with the definition of prochirality.

5.2.2. Homotropicity

The $S_9^{[4]}(/S_3^{[4]})$-orbit for the allene skeleton (**1**) is determined to be homotropic, since both $S_9^{[4]}$ and $S_3^{[4]}$ are astereogenic. The "stereogenicity fittingness" column of table 4 shows that the $S_9^{[4]}(/S_3^{[4]})$-orbit accommodates four achiral or chiral ligands of the same kind. They are depicted in figure 12. From a stereochemical (point-group

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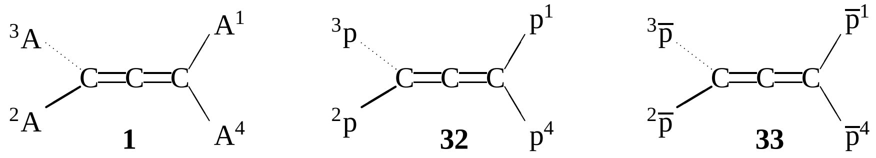
[19] The stem "tropic" of the terms *homotropic, enantiotropic,* and *hemitropic* comes from the Greek "trópos" (turning).

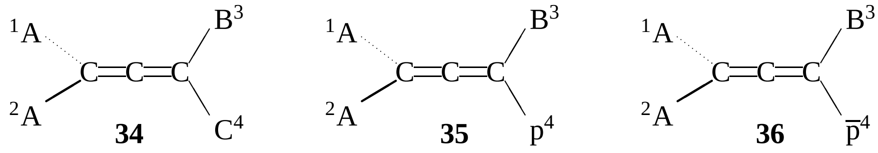


1
$D_{2d}(/C_s)$
homospheric
$S_9^{[4]}(/S_3^{[4]})$
homotropic

32
$D_2(/C_1)$
hemispheric
$S_9^{[4]}(/S_3^{[4]})$
homotropic

33
$D_2(/C_1)$
hemispheric
$S_9^{[4]}(/S_3^{[4]})$
homotropic

Figure 12. Allenes with A_4 and p_4 ($\overline{p}_4$).



34
$C_s(/C_1) + 2C_s(/C_s)$
$C_s(/C_1)$: enantiospheric
$S_3^{[4]}(/S_1^{[4]}) + 2S_3^{[4]}(/S_3^{[4]})$
$S_3^{[4]}(/S_1^{[4]})$: enantiotropic

35
$4C_1(/C_1)$
$C_1(/C_1)$: hemispheric
$S_3^{[4]}(/S_1^{[4]}) + 2S_3^{[4]}(/S_3^{[4]})$
$S_3^{[4]}(/S_1^{[4]})$: enantiotropic

36
$4C_1(/C_1)$
$C_1(/C_1)$: hemispheric
$S_3^{[4]}(/S_1^{[4]}) + 2S_3^{[4]}(/S_3^{[4]})$
$S_3^{[4]}(/S_1^{[4]})$: enantiotropic

Figure 13. Allenes with A_2BC and A_2Bp ($A_2B\overline{p}$).

theoretical) point of view, the homospheric $D_{2d}(/C_s)$-orbit in **1** is desymmetrized into the hemispheric $D_2(/C_1)$-orbit in the allene derivative **32** (or **33**). From the permutation-group theoretical point of view, however, the homotropic $S_9^{[4]}(/S_3^{[4]})$-orbit in **1** remains invariant so that the allene derivative **32** (or **33**) has the same homotropic $S_9^{[4]}(/S_3^{[4]})$-orbit.

5.2.3. Enantiotropicity

Let us next consider an allene derivative (**34**) with A_2BC, which belongs to the point group C_s and to the permutation group $S_3^{[4]}$. As shown in figure 13, the two A's placed on the positions 1 and 2 construct an enantiospheric orbit governed by $C_s(/C_1)$. The two A's on the positions 1 and 2 alternatively belong to an orbit of permutational type, i.e., $S_3^{[4]}(/S_1^{[4]})$, which is enantiotropic in terms of the criterion listed in table 4. Hence, **34** is prochiral because of the enantiospheric $C_s(/C_1)$-orbit; at the same time, it is prostereogenic because of the enantiotropic $S_3^{[4]}(/S_1^{[4]})$-orbit.

From a permutational point of view, **35** (or **36**) is characterized by the same CR, as listed in figure 13. This fact is in agreement with the criterion listed in table 4. On the other hand, **35** (or **36**) belongs to the point group C_1 so that the four ligands (A, A, B, and p) are nonequivalent from a stereochemical point of view, giving four one-membered $C_1(/C_1)$-orbits distinctly.

5.2.4. Hemitropicity

The allene derivative **14** belongs to the point group C'_2, where two A's and two B's separately construct two-membered hemispheric orbits governed by the CR $C'_2(/C_1)$. At the same time, **14** belongs to the permutation group $S^{[4]}_{2'}$, where the two A's and the two B's separately two-membered hemitropic orbits governed by the CR $S^{[4]}_{2'}(/S^{[4]}_1)$. The allene derivative **14** is stereogenic because of the global permutation symmetry $S^{[4]}_{2'}$ (a stereogenic group) in the light of the criterion listed in table 2. This means the existence of a diastereomer **15**, which is also enantiomeric to **14** from a stereochemical point of view. It follows that the diastereomeric relationship and the enantiomeric relationship overlap in the allene derivative **14**.

Such an overlap is resolved in the allene derivative **23**, which belongs to the point group C'_2. Thus, the two A's and the two p's separately construct two-membered hemispheric orbits governed by the CR $C'_2(/C_1)$. The orbit of the two A's in **23** coincides with the orbit of the two A's in the corresponding enantiomer $\overline{\mathbf{23}}$, as found in figure 6. This behavior is characteristic of such a hemispheric $C'_2(/C_1)$-orbit.

At the same time, **23** belongs to the permutation group $S^{[4]}_{2'}$, where two A's and two p's separately construct two-membered hemitropic orbit governed by the CR $S^{[4]}_{2'}(/S^{[4]}_1)$. The orbit of the two A's in **23** coincides with the orbit of the two A's in the corresponding diastereomer **24** under permutation-group symmetry, as found in figure 7. The behavior between such diastereomers is characteristic of a hemitropic orbit such as the $S^{[4]}_{2'}(/S^{[4]}_1)$-orbit. The allene derivative **23** is stereogenic because of the global permutation symmetry $S^{[4]}_{2'}$ (a stereogenic group), which means the existence of a diastereomer **24**.

5.3. Desymmetrization and subduction

From a stereochemical (point-group theoretical) point of view, a derivation from a given skeleton is explained by the subduction of a CR [16,18]. Thus, each skeleton is desymmetrized into an appropriate subgroup, where the original CR is subduced into a sum of CRs according to the precalculated subduction pattern of the original CR. The resulting CRs in the sum correspond to orbits of the subgroup, each of which accommodates a set of ligands in terms of chirality fittingness that is controlled by the sphericity of the orbit, as shown in table 3 [16]. As for the derivation of **14** belonging to C'_2, the four positions of the allene skeleton (**1**) are divided into two $C'_2(/C_1)$-orbits in agreement with the subduction represented by

$$D_{2d}(/C_s) \downarrow C'_2 = 2C'_2(/C_1). \tag{6}$$

The resulting orbits $\{1,2\}$ and $\{3,4\}$ are respectively governed by the CR $C'_2(/C_1)$. It should be noted that we have

$$C'_2 = \{I, C_{2(1)}\} \sim \big\{(1)(2)(3)(4), (1\ 2)(3\ 4)\big\}, \tag{7}$$

$$C'_2(/C_1) = \big\{(1)(2), (1\ 2)\big\} = \big\{(3)(4), (3\ 4)\big\}. \tag{8}$$

Since both the global symmetry C'_2 and the local symmetry C_1 are chiral, the CR $C'_2(/C_1)$ is determined to be hemispheric. According to the hemisphericity, the $C'_2(/C_1)$-orbit can accommodate two achiral ligands of the same kind (A or B) or two chiral ligands of the same kind (p or $\overline{\mathrm{p}}$). Thereby, we obtain a C'_2-molecule (**14** or **15**) of formulas A_2B_2 as well as another C'_2-molecule (**23** or **24**) of formulas A_2p_2.

As found easily in the discussion described above, we can consider the subduction of a CR of permutation type, i.e., $P(/P_i)$. Thus, each skeleton is desymmetrized into an appropriate subgroup, where the original CR is subduced into a sum of CRs according to the precalculated subduction pattern of the original CR. The resulting CRs in the sum correspond to orbits of the subgroup, each of which is controlled as collected in table 4. When we take account of permutation symmetry in the derivation of **14** belonging to $S^{[4]}_{2'}$, the four positions of the allene skeleton (**1**) are divided into two orbits $\{1,2\}$ and $\{3,4\}$ according to the subduction:

$$S^{[4]}_{9}\big(/S^{[4]}_{3}\big) \downarrow S^{[4]}_{2'} = 2S^{[4]}_{2'}\big(/S^{[4]}_{1}\big), \tag{9}$$

where the orbits are respectively governed by the CR $S^{[4]}_{2'}(/S^{[4]}_{1})$. It should be noted that we have

$$S^{[4]}_{2'} = \big\{(1)(2)(3)(4), (1\ 2)(3\ 4)\big\}, \tag{10}$$

$$S^{[4]}_{2'}(/S^{[4]}_{1}) = \big\{(1)(2), (1\ 2)\big\} = \big\{(3)(4), (3\ 4)\big\}. \tag{11}$$

Since both $S^{[4]}_{2'}$ and $S^{[4]}_{1}$ are stereogenic, the CR $S^{[4]}_{2'}(/S^{[4]}_{1})$ is determined to be hemitropic. The hemitropicity controls the modes of accommodation that rationalize the generation of the diastereomers **14** and **15** as well as the generation of the diastereomers **23** and **24**. Note that the diastereomeric relationship between **14** and **15** overlaps the enantiomeric relationship.

5.4. Prostereogenicity

As defined above, prostereogenicity is ascribed to the presence of at least one enantiotropic orbit. For example, the two A's on the positions 1 and 2 in the allene derivative **34** belong to an enantiotropic orbit governed by the CR $S^{[4]}_{3}(/S^{[4]}_{1})$. The prostereogenic nature of the two A's can be characterized by the subduction into the maximal stereogenic subgroup, which is $S^{[4]}_{1}$ in this case:

$$S^{[4]}_{3}(/S^{[4]}_{1}) \downarrow S^{[4]}_{1} = 2S^{[4]}_{1}(/S^{[4]}_{1}). \tag{12}$$

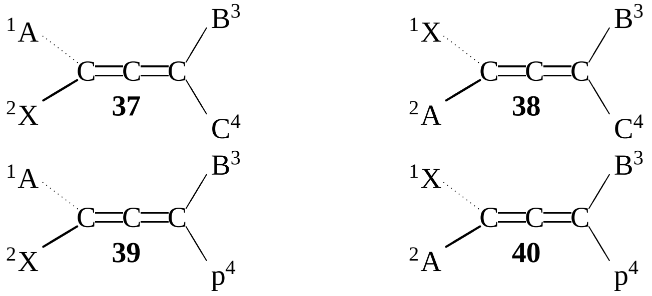
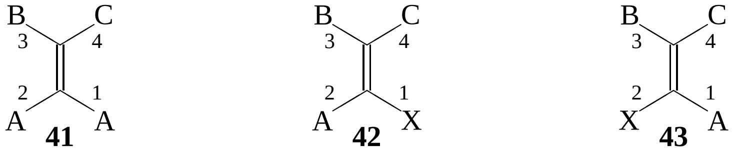


Figure 14. Derivation of allenes with A_2BC and A_2Bp ($A_2B\overline{p}$).



$4C_s(/C_s)$	$4C_s(/C_s)$	$4C_s(/C_s)$
A+A+B+C	A+X+B+C	A+X+B+C
$S_3^{[4]}(/S_1^{[4]}) + 2S_3^{[4]}(/S_3^{[4]})$	$4S_1^{[4]}(/S_1^{[4]})$	$4S_1^{[4]}(/S_1^{[4]})$
A_2+B+C	A+X+B+C	A+X+B+C

Figure 15. Prostereogenicity in an ethylene with A_2BC. The replacement of respective A's in **41** produces **42** and **43**.

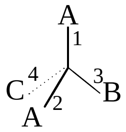
The permutation-group desymmetrization represented by equation (12) produces a duplex of diastereomers (**37** and **38**), as exemplified in the top row of figure 14. Note that the diastereomeric relationship and the enantiomeric relationship overlap in this case.

On the other hand, the subduction represented by equation (12) also explains the desymmetrization of the two A's in **35**. Each one of the two A's is replaced by X to produce one of **39** and **40**, which are diastereomeric to each other.

Prostereogenicity in an ethylene with A_2BC is illustrated in figure 15. The ethylene **41** belongs to the point group C_s (achiral) as well as to the permutation group $S_3^{[4]}$ (astereogenic). The prostereogenicity of **41** stems from the enantiotropic $S_3^{[4]}(/S_1^{[4]})$-orbit accommodating two A's. The permutational desymmetrization of **41** is controlled by the subduction represented by equation (12) so as to produce **42** and **43**.

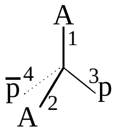
6. Stereogenicity versus chirality

The $D_{2d}(/C_2')$-orbit described in figure 6 accommodates the four sets of molecules, i.e., {**23a**, **23b**}, {**23c**, **23d**}, {$\overline{\mathbf{23}}$**a**, $\overline{\mathbf{23}}$**b**}, and {$\overline{\mathbf{23}}$**c**, $\overline{\mathbf{23}}$**d**}. The former two are homomeric and the latter two are enantiomeric to the original **23**. Thus, the mode of accommodation is determined to be a compensated chiral packing in accord with chirality fittingness shown in table 3, since the $D_{2d}(/C_2')$-orbit is enantiospheric.



44

$C_s(/C_1) + 2C_s(/C_s)$
$A_2 + B + C$
prochiral
$S_3^{[4]}(/S_1^{[4]}) + 2S_3^{[4]}(/S_3^{[4]})$
$A_2 + B + C$
prostereogenic



45

$C_s(/C_1) + 2C_s(/C_s)$
$p\overline{p} + A + A$
prochiral
$S_3^{[4]}(/S_1^{[4]}) + 2S_3^{[4]}(/S_3^{[4]})$
$A_2 + p + \overline{p}$
prostereogenic

Figure 16. Prostereogenicity and prochirality of tetrahedral molecules with A_2BC and $A_2p\overline{p}$.

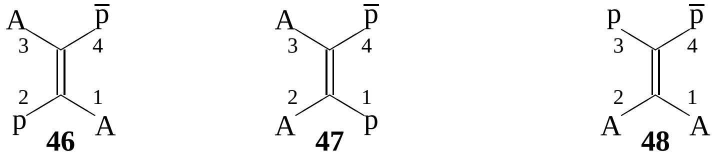
The parallelism between point-group and permutation-group symmetries is effective to explain such modes of accommodation. The $S_9^{[4]}(/S_{2'}^{[4]})$-orbit described in figure 7 accommodates the four sets of derivatives, i.e., {**23a**, **23b**}, {**23c**, **23d**}, {**24a**, **24b**}, and {**24c**, **24d**}. The former two are homomeric and the latter two are diastereomeric to the original **23**. The enantiotropicity of the $S_9^{[4]}(/S_{2'}^{[4]})$-orbit permits us to consider "stereogenicity fittingness" on the analogy of chirality fittingness. Thus, the mode of accommodation is regarded as a *compensated stereogenic packing* for the enantiotropic orbit.

As chirality is concerned with the symmetry of a molecule, stereogenicity is concerned with the permutability of a molecule. It should be emphasized that the concept of stereogenicity in this context specifies the global nature of the molecule. As we take account of local (point-group) symmetry and local chirality, we should consider local permutation-group symmetry and local stereogenicity. The new definition of stereogenicity proposed in this paper is conceptually different from the original "stereogenicity" [10] in that global and local stereogenicities are strictly differentiated after stereogenic and astereogenic permutation groups are defined.

7. Prostereogenicity versus prochirality

Prostereogenicity and prochirality of a tetrahedral molecule with A_2BC (**44**) are illustrated in figure 16. The enantiospheric $C_s(/C_1)$-orbit accommodates A_2 so as to show prochirality. At the same time, the A_2 belong to the enantiotropic $S_3^{[4]}(/S_1^{[4]})$-orbit, which indicates the prostereogenicity of **44**.

On the other hand, a tetrahedral molecule with $A_2p\overline{p}$ has an enantiospheric $C_s(/C_1)$-orbit, which accommodates a pair of p and $\overline{p}$ to give a compensated chiral packing. The two A's are chemically different and distinctly occupy one-membered $C_s(/C_s)$-orbits. Under permutation-group symmetry, the A_2 belong to the enantiotropic $S_3^{[4]}(/S_1^{[4]})$-orbit, which indicates the prostereogenicity of **45**. The dif-



46	**47**	**48**
$2C_i(/C_1)$	$2C'_s(/C_1)$	$2C_s(/C_1)$
$A_2 + p\overline{p}$	$A_2 + p\overline{p}$	$A_2 + p\overline{p}$
prochiral	prochiral	prochiral
$4S_1^{[4]}(/S_1^{[4]})$	$4S_1^{[4]}(/S_1^{[4]})$	$S_3^{[4]}(/S_1^{[4]}) + 2S_3^{[4]}(/S_3^{[4]})$
$A + A + p + \overline{p}$	$A + A + p + \overline{p}$	$A_2 + p + \overline{p}$
stereogenic	stereogenic	prostereogenic

Figure 17. Stereogenicity, prostereogenicity and prochirality of ethylene derivatives with $A_2p\overline{p}$.

ference between the modes of accommodation in **45** shows that point-group and permutation-group symmetries should be discussed distinctly to avoid undesired confusion.[20]

Although both **44** and **45** are decided to be prochiral and prostereogenic, the sources of their properties are different. This set of examples indicates the importance of orbits under point-group and permutation-group symmetries, rather than so-called "prochiral centers" or "prostereogenic centers." This means that *prostereogenic nature should be ascribed to the enantiotropicity of an orbit at issue*, whereas it has been conventionally ascribed to such a prostereogenic center or unit [12].

The ethylene derivative **46** is prochiral because of its two enantiospheric $C_i(/C_1)$-orbits accommodating two A's and a pair of p and $\overline{p}$, respectively. The ethylene derivative **47** is also prochiral because of its two enantiospheric $C'_s(/C_1)$-orbit. Since they distinctly belong to a stereogenic group $S_1^{[4]}$, they are decided to be stereogenic so as to show diastereomeric relationship. In fact, they are converted into each other to give a duplex of diastereomers under the permutation group $S_9^{[4]}$.

The ethylene derivative **48** is constitutionally isomeric to the duplex of diastereomers (**46** and **47**). It is prochiral because the two A's occupy an enantiomeric $C_s(/C_1)$-orbit and a pair of p and $\overline{p}$ occupies another enantiomeric $C_s(/C_1)$-orbit. Since the two A's are alternatively considered to occupy an enantiotropic $S_3^{[4]}(/S_1^{[4]})$-orbit, it is decided to be prostereogenic. Note that p and $\overline{p}$ accommodated in the $C_s(/C_1)$-orbit are equivalent under point-group symmetry, whereas they are non-equivalent under permutation symmetry because of the CRs $2S_3^{[4]}(/S_3^{[4]})$. This situation of permutation-group symmetry is essentially equivalent to the case of **41**, although the participating ligands are different in chirality/achirality.

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[20] From the present viewpoint, Hanson's term "prochiral" [9] corresponds to the accommodation $A^2+p+\overline{p}$ shown in figure 16. Hence, it should be replaced by the term "prostereogenic".

8. Occasional overlap of stereogenicity and prostereogenicity

Whereas a chiral molecule cannot be prochiral, a stereogenic molecule can be prostereogenic as discussed for the square-planar complex with A_2BC in figure 11. The discussion described above can be elaborated into a more systematic format in the light of global/local stereogenicity under permutation-group symmetry. Note that $S^{[4]}$ is represented by

$$S^{[4]} = S_9^{[4]} + \pi_1 S_9^{[4]} + \pi_2 S_9^{[4]}, \tag{13}$$

where $\pi_I = (1)(2)(3)(4)$ (abbreviated), $\pi_1 = (1\ 2)(3)(4)$ and $\pi_2 = (1)(2\ 3)(4)$ are selected as the representatives of the three cosets. The three cosets respectively correspond to the modes of numbering shown for the complexes **11**, **12**, and **13** (figure 3). As a result, these complexes are interconvertible under the permutation group $S^{[4]}$. The permutation groups (stabilizers) for these modes of numbering are calculated to be $S_9^{[4]}$ for **11**, $\pi_1 S_9^{[4]} \pi_1^{-1}$ for **12**, and $\pi_2 S_9^{[4]} \pi_2^{-1}$ for **13**, which are a set of maximal stereogenic subgroups conjugate to each other under $S^{[4]}$.

The permutation group $S_3^{[4]} = \{\pi_I, \pi_1\} = \{(1)(2)(3)(4), (1\ 2)(3)(4)\}$ keeps **11** invariant, because the resulting complex **12** is equalified to the original complex **11**. This fact indicates that **11** belongs to $S_3^{[4]}$ on the action of $S^{[4]}$. Since $\pi_1 (= (1\ 2)(3)(4))$ is not contained in $S_9^{[4]}$, the permutation group $S_3^{[4]}$ should be regarded as being astereogenic for this case. On the other hand, the same group $S_3^{[4]}$ keeps **13** invariant, where the numbering is altered so that the same assignment of A_2BC gives the complex **13**. Note that the numbering is generated by applying π_2 $(= (1)(2\ 3)(4))$ to the original numbering of **11**. Thereby, the complex **13** with the altered numbering is determined to belong to the group $S_3^{[4]}$. As found by a simple calculation, the group $S_3^{[4]}$ is a subgroup of the permutation group $\pi_2 S_9^{[4]} \pi_2^{-1}$, which is conjugate to $S_9^{[4]}$ under $S^{[4]}$. Since π_1 $(= (1\ 2)(3)(4))$ is contained in $\pi_2 S_9^{[4]} \pi_2^{-1}$, the permutation group $S_3^{[4]}$ should be regarded as being stereogenic for this case. It follows that, although the same permutation group $S_3^{[4]} = \{\pi_I, \pi_1\}$ is ascribed to **11** (or **12**) and **13**, the respective actions are different in stereogenicity.

By virtue of the astereogenicity of the permutation group $S_3^{[4]}$ for **11** (or **12**), the $S_3^{[4]}(/S_1^{[4]})$-orbit accommodating the two A's is enantiotropic. This means that **11** (or **12**) is prostereogenic with respect of the orbit.

On the other hand, since the same permutation group $S_3^{[4]}$ is stereogenic for **13**, the $S_3^{[4]}(/S_1^{[4]})$-orbit accommodating the two A's is hemitropic. The pairwise counterpart of the $S_3^{[4]}(/S_1^{[4]})$-orbit of **13** is found to be the $S_3^{[4]}(/S_1^{[4]})$-orbit of **11** (or **12**), because the group $S^{[4]}$ equalifies **11**, **12**, and **13** as diastereomers.

9. Features of the CIP system

In the light of the discussion described above, stereogenicity/astereogenicity has been ascribed to permutation-group symmetry, insomuch that chirality/achirality has been based on point-group symmetry (table 1). It is worthwhile to discuss the CIP system from the standpoint of our approach.

The standpoint of the CIP system was described in Prelog's Nobel Lecture [34], which showed a catalog listing 31 chiral models as figure 13. In this catalog, enantiomeric and diastereomeric relationships are discussed on the same level in terms of the criterion of permutability. Such enantiomeric relationship as Prelog claimed on the basis of the permutability, however, should be regarded as diastereomeric relationship by virtue of the present formulation. Although the R- and S-symbols of the CIP system originally aimed at specifying enantiomeric relationship, they have been used to differentiate between ABCp and ACBp, which are not enantiomeric but diastereomeric. If the R- and S-symbols specify enantiomeric relationship, ABCp and ACB$\overline{\mathrm{p}}$ should be discriminated. Even if the symbols are used to specify ABCD and ABDC, they are considered to specify the diastereomeric relationship overlapped with the enantiomeric relationship.[21] They are not directly concerned with the enantiomeric relationship.

In summary, *the CIP system specifies diastereomeric relationship but does not specify enantiomeric relationship*. Indeed, the CIP system is based on the fact that diastereomeric relationship overlaps enantiomeric relationship in allene and tetrahedral skeletons. As found in figure 8, the permutations in the $D_2\sigma_{\mathrm{d}(1)}$-part for an allene skeleton are identical with those contained in $(1\ 4)(2)(3)S_7^{[4]}$-part without considering overlines, i.e., chemically speaking, with considering achiral ligands only. For a tetrahedral skeleton, the permutations in the $T\sigma_{\mathrm{d}(1)}$-part are identical with those in $(1)(2\ 3)(4)S_{10}^{[4]}$-part without considering overlines. This is the reason to discuss "chirality" and "stereogenicity" for a tetrahedral model in terms of the observance and violation of chirality fittingness [30]. Of course, the points described in the present paper do not spoil the practical versatility of the CIP system. Rather, they bring a sound logical framework to the CIP system particularly by differentiating enantiomeric relationships from diastereomeric relationships.

The feature of the CIP system becomes clearer by examining the diastereomeric relationship between the tetrahedral derivatives **5** and **6** (figure 2), which are conventionally regarded as having pseudo-asymmetric centers. From the present point of view, each of them belongs to $S_1^{[4]}$, which is determined to be stereogenic under the action of $S^{[4]}$ (table 2). The stereogenicity is the source of the diastereomeric relationship between **5** and **6**, which is specified by the symbols r and s of the CIP system.

The square-planar complexes **8**, **9**, and **10** are determined to be diastereomeric under $S^{[4]}$. Each of them belongs to the permutationg group $S_1^{[4]}$, which is determined to be stereogenic (table 2). The diastereomeric relationship between **8** and **9** overlaps their enantiomeric relationship.

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[21] It is convenient to classify stereogenic molecules by considering such overlap of enantiomeric and diastereomeric relationships. If a molecule is converted under a permutation-group symmetry into its diastereomer that is identical with its enantiomer, it is defined as being *enantiostereogenic* (or shortly, enantiogenic); otherwise, it is defined as being *diastereogenic*.

10. Conclusion

By using molecules of ligancy 4 that have been derived from an allene, an ethylene, a tetrahedral, and a square-planar skeleton, the concept of stereogenicity has been investigated in detail. Thus, enantiomeric relationship has been discussed in terms of an orbit generated by point-group symmetry, while diastereomeric relationship has been considered to be concerned with an orbit under permutation-group symmetry. After stereoisomeric groups were proposed to integrate point groups and permutation groups, the permutation groups have been examined as subgroups of such stereoisomeric groups. Proper and improper permutations have been defined to classify the permutation groups into stereogenic and astereogenic ones. The nature of being enantiomeric has been ascribed to chirality due to point-group symmetry, while the nature of being diastereomeric has been ascribed to stereogenicity due to permutation-group symmetry. The global and local stereogenicities (or astereogenicities) of an orbit that is governed by a coset representation under permutation-group symmetry have been examined to define the tropicity of the orbit, where the terms *homotropic*, *enantiotropic*, and *hemitropic* are coined. The derivation of the skeletons has been characterized by desymmetrization due to the subduction of coset representations. Relevant subjects such as prostereogenicity and the CIP system have been disscussed for restructuring stereochemistry and stereoisomerism.

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